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1989 J. Phys. A: Math. Gen. 22 L143

(http://iopscience.iop.org/0305-4470/22/5/002)

J. Phys. A: Math. Gen. **22** (1989) L143-L147. Printed in the UK

LETTER TO THE EDITOR

Solvability in quantum mechanics and classically superfluous invariants

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Received 1 October 1988

Abstract. We show that in quantum mechanics there can be more commuting constants of motion than the dimension suggests. There will be an algebraic relationship between the invariants, but when this relationship is non-trivial it can be used to reduce the Schrödinger equation to quadratures. The Lamé equation and some of its generalisations fall into the class of potentials for which this method works.

The usual definition of (Liouville) integrability of an N-dimensional Hamiltonian system requires the existence of N constants of motion (the Hamiltonian being one of them) in involution. This definition is algebraic and extends naturally to quantum mechanics; there we just work with operators that should commute (Hietarinta 1984).

In classical mechanics Liouville integrability means that the system can in principle be solved in quadratures (Arnold 1980); however, in practice this is not so straightforward. The situation is even worse in quantum mechanics. According to the above definition all one-dimensional systems are trivially integrable since the Hamiltonian itself is sufficient for the one required constant of motion. In classical mechanics a solution in quadratures can indeed be easily written down. However, in quantum mechanics the one-dimensional Schrödinger equation is usually far from solvable.

In this letter we introduce an extension of integrability which is related to solvability. By solvability we mean that the Schrödinger equation should be solvable by quadratures. The idea is essentially to construct more than N constants of motion. This concept works only in quantum mechanics due to the non-commuting nature of its variables and is trivial in classical mechanics. (In more mathematical terms, we use the fact that the operators in quantum mechanics do not factorise uniquely.) Of course, when there are more than N integrals of motion they cannot all be functionally independent, but in quantum mechanics such extra constants can nevertheless be used to integrate the Schrödinger equation.

Let us consider the one-dimensional Hamiltonian

$$H = -\tfrac{1}{2}\hbar^2\partial_x^2 + V(x). \tag{1}$$

We want to find an operator I which commutes with H. Since the highest derivative in H has a constant coefficient this property must hold also for I. We can always subtract powers of H from I, therefore I must start like ∂_x^{2n+1}. Thus the simplest self-adjoint operator that can produce something useful is

$$I_3 = -\mathrm{i}\hbar^3\partial_x^3 + \mathrm{i}\tfrac{1}{2}\hbar(\partial_x C(x) + C(x)\partial_x). \tag{2}$$

0305-4470/89/050143+05$02.50

The condition $[H, I_3]=0$ gives the following equations for C and V:

$$C'-3V'=0 \tag{3a}$$

$$\hbar^2(C'''-4V''')+4CV'=0. \tag{3b}$$

From (3a) we get $C=3V+c_1$, and when this is substituted into (3b) we get

$$\hbar^2V'''+4(3V+c_1)V'=0. \tag{4}$$

This can be easily integrated twice, and if we introduce the function $\mathcal{P}$ by

$$V=\hbar^2\mathcal{P}(x)-c_1 \qquad C=3\hbar^2\mathcal{P}(x) \tag{5}$$

the twice-integrated form of (4) becomes one of the defining equations for Weierstrass' $\mathcal{P}$ function (Abramowitz and Stegun 1972):

$$\mathcal{P}'(x)^2=4\mathcal{P}(x)^3-g_2\mathcal{P}(x)-g_3. \tag{6}$$

Note the factor $\hbar^2$ in the potential, which again shows the purely quantum mechanical nature of this result. The two parameters g_i together with c_1 and translation in x, amount to four free constants, which is what we should have according to (3). The constant c_1 can be absorbed into the energy so we will ignore it from now on.

Thus if $V=\hbar^2\mathcal{P}(x)$ there exists an operator, third order in ∂_x, which commutes with the Hamiltonian. What can we do with this information? Since we have two commuting operators they have simultaneous eigenfunctions

$$(-\tfrac{1}{2}\hbar^2\partial_x^2+V(x))\Psi(x)=E\Psi(x) \tag{7}$$

$$[-\mathrm{i}\hbar^3\partial_x^3+\mathrm{i}\tfrac{1}{2}\hbar(\partial_xC(x)+C(x)\partial_x)]\Psi(x)=K\Psi(x) \tag{8}$$

where E and K are the respective eigenvalues. Equation (8) can now be reduced to a *linear* equation in ∂_x: if we solve $\partial_x^2\Psi$ from (7) and substitute into (8), we obtain

$$\frac{\Psi'}{\Psi}=\frac{\mathcal{P}'-2\mathrm{i}\hbar^{-3}K}{2(\mathcal{P}+2\hbar^{-2}E)}. \tag{9}$$

Before integrating (9) let us see if it is compatible with (7)—note that (7) contains only E while both E and K appear in (9). In fact when (7) is calculated using (9) for Ψ' we get the following condition between the parameters:

$$-4\hbar^{-6}K^2=-32\hbar^{-6}E^3+2\hbar^{-2}Eg_2-g_3. \tag{10}$$

The relationship (10) allows a natural parametrisation of E and K. For a given E define α by

$$E=-\tfrac{1}{2}\hbar^2\mathcal{P}(\alpha). \tag{11a}$$

Then by virtue of (10) and (6)

$$K=\pm\mathrm{i}\tfrac{1}{2}\hbar^3\mathcal{P}'(\alpha) \tag{11b}$$

and (9) takes the particularly nice form

$$\frac{\Psi'(x)}{\Psi(x)}=\frac{\mathcal{P}'(x)\pm\mathcal{P}'(\alpha)}{2(\mathcal{P}(x)-\mathcal{P}(\alpha))}. \tag{12}$$

Ψ can now be integrated and the result expressed in terms of Weierstrass' $\mathcal{P}$, σ and ζ functions (Whittaker and Watson 1927). However, that solution involves additional

transcendental functions, so we prefer to write the solution in quadratures, as promised before. Using (12) and (6) we find

$$\Psi(x)=(\mathscr{P}(x)-\mathscr{P}(\alpha))^{1/2}\exp\left(\pm\tfrac{1}{2}\mathscr{P}'(\alpha)\int^{\mathscr{P}(x)}(4t^3-g_2t-g_3)^{-1/2}\,\mathrm{d}t\right). \tag{13}$$

The $\hbar^2$ dependence would enter if we wrote this in terms of V instead of $\mathscr{P}$. Here energy dependence enters implicitly through α given by (11). The result (13) is formal and in applications one should check whether $\mathscr{P}'(\alpha)$ is real (bounded states) or pure imaginary (oscillatory states) and, in the former case, whether the state in normalisable. These questions will not be discussed here.

The above steps can be carried through for higher-order invariants as well. As a result one obtains higher-order equations for the potential $V(x)$. To simplify the notation let $\partial=\partial_x$, and let us also define ε and v by

$$E=\tfrac{1}{2}\hbar^2\varepsilon \qquad V(x)=\tfrac{1}{2}\hbar^2 v(x). \tag{14}$$

A fifth-order generalisation of the third-order result would start with

$$I_5=\mathrm{i}\partial^5-\mathrm{i}\tfrac{1}{2}(\partial^3B(x)+B(x)\partial^3)+\mathrm{i}\tfrac{1}{2}(\partial D(x)+D(x)\partial). \tag{15}$$

The condition $[H, I_5]=0$ yields the equations

$$\begin{aligned} &B'-\tfrac{5}{2}v'=0 \qquad 9B'''-4D'-20v'''+6v'B=0\\ &B^{(5)}+3B''v'+3B'v''-D'''-2v^{(5)}+2v'''B-2v'D=0. \end{aligned} \tag{16}$$

The first two equations yield B and D in terms of v

$$B=\tfrac{5}{2}v \qquad D=\tfrac{5}{8}(v''+3v^2)+d_0 \tag{17}$$

and then what remains is a fifth-order equation for v:

$$v^{(5)}-10v'''v-20v''v'+30v'v^2+16v'd_0=0. \tag{18}$$

At this point we should note that this is a more general result than that obtained from taking $I_5=HI_3+I_3H$, which only yields a particular solution.

In the above derivation of (18) we used the condition $[H, I]=0$. As an alternative let us instead use the fact that H and I_5 must have simultaneous eigenfunctions. From $H\Psi=\varepsilon\Psi$ we get for the second derivative $\Psi''=(v-\varepsilon)\Psi$ and then from $I_5\Psi=K\Psi$ we obtain (assuming (17)) the following expression for Ψ'/Ψ:

$$\frac{\Psi'}{\Psi}=\frac{v'''-6v'v-4v\varepsilon+\mathrm{i}16K}{2(v''-3v^2-4v\varepsilon-8\varepsilon^2-8d_0)}. \tag{19}$$

When this is substituted back into $\Psi''=(v-\varepsilon)\Psi$ to check compatibility, we get a lengthy expression, which is, in particular, polynomial in ε and K. Anticipating the analogue of (10) we may expand

$$-K^2=k_5\varepsilon^5+k_4\varepsilon^5+k_4\varepsilon^4+k_3\varepsilon^3+k_2\varepsilon^2+k_1\varepsilon+k_0. \tag{20}$$

Now since v, B and D do not depend on ε the coefficients of various powers of ε must be constants. In this way we obtain from the coefficients of descending powers of ε first that $k_5=-1$, $k_4=0$, $k_3=-2d_0$ (assuming the results (17)). From ε^2 we get the *once-integrated* form of (18)

$$v''''-10v''v-5v'^2+10v^3+16vd_0-16k_2=0. \tag{21}$$

The surprising fact is that (21) can be integrated further by the remaining coefficients of ε. If (21) is implemented, then we get

$$W_3 \equiv 2v'''v' - v''^2 - 10v'^2v + 5v^4 + 16d_0v^2 - 32k_2v + 64(k_1 + d_0^2) = 0. \tag{22}$$

It is easy to see that (21) is a differential consequence of (22), and what is interesting is that this twice-integrated form of (18) came for free. The trick works one step further: if we implement (22) then the ε-independent part yields ($k_{10} = k_1 + d_0^2$)

$$\begin{aligned} W_2 \equiv\ & v''^4 - 20v''^2v'^2v - 10v''^2v^4 - 32v''^2v^2d_0 + 64v''^2vk_2 - 128v''^2k_{10} + 8v''v^4 \\ & + 80v''v'^2v^3 + 128v''v'^2vd_0 - 128v''v'^2k_2 - 20v'^4v^2 - 64v'^4d_0 - 76v'^2v^5 \\ & - 192v'^2v^3d_0 + 256v'^2v^2k_2 + 256v'^2vk_{10} - 1024v'^2k_0 + 1024v'^2k_2d_0 \\ & + 25v^8 + 160v^6 - 320v^5k_2 + 640v^4k_{10} + 246v^4d_0^2 - 1024v^3k_2d_0 \\ & + 1024v^2(2k_{10}d_0 + k_2^2) - 4096vk_{10}k_2 + 4096k_{10}^2 = 0. \end{aligned} \tag{23}$$

However, this is *not* equivalent to (22). If we divide (23) by v'^2 (to make k_0 a bare constant) and differentiate it we get

$$(W_2/v'^2)' = W_3F. \tag{24}$$

The problem is that the integrating factor F above can vanish for non-trivial functions v that are not solutions of (22). This means that although (23) contains lower-degree derivatives of v, and is in that sense better than (22), it also allows spurious solutions, and in that sense the result (23) is not satisfactory. For the fifth-order case our final equation for v is therefore (22).

It is easy to check that the above equation (22) for v allows the solutions $v = 2\mathcal{P}$ and $6\mathcal{P}$, but since (22) contains more constants it also allows other solutions. We may consider (22) as defining a generalisation of the Weierstrass $\mathcal{P}$ function.

The method above can be extended to still higher-order I and will lead to still more general equations for the potential v. For Ψ'/Ψ we will obtain equations of the form

$$\frac{\Psi'}{\Psi} = \frac{J'(x,\varepsilon) \pm \mathrm{i}K(\varepsilon)}{2J(x,\varepsilon)} \tag{25}$$

where $K(\varepsilon)$ is given by a generalisation of (20). The potential v can always be integrated a few times by the requirement of compatibility with $\Psi'' = (v - \varepsilon)\Psi$, which leads to

$$2J''J - J'^2 - K^2 - 4J^2(v - \varepsilon) = 0. \tag{26}$$

The above examples showed that this equation has solutions where J is linear or quadratic in ε and K^2 is third or fifth order, respectively. It is reasonable to assume that (26) also has higher-order solutions. For example if I is an operator that commutes with H then so does $IH + HI$, and this last one will generate a higher-order J. In general if J is of degree n then K^2 should be of order $2n+1$. Note that it is K^2 that is polynomial in ε, not K.

Higher-order polynomial solutions provide us with more general potentials for which this procedure works. Lower-order results suggest that this will give us at least the Lamé equations with $v = n(n+1)\mathcal{P}(x)$. But since the defining equation for v will be of higher and higher order, we will also get a genuinely wider class of potentials.

We have considered one-dimensional Hamiltonians that have a commuting operator which is odd in momenta. In this way we have obtained a class of potentials for which Ψ'/Ψ is a rational function of ε, v and its derivatives. This allows us to express Ψ in quadratures in terms of v, v', ... which may be considered as known. Probably this method can be extended to other kinds of systems.

I would like to thank B Grammaticos for discussions. This work was supported by the Academy of Finland.

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